REFERENCES

1. GIRS I. V. and SRETENSKII L. N., Effect of a change in the principal dimensions of a ship on its wave resistance. *Prikl. Mat. Mekh.* **10,** 1, 21–32, 1946.
2. SRETENSKII L. N., *Theory of the Wave Motions of a Fluid.* Nauka, Moscow, 1977.
3. *Handbook of Ship Theory*, Vol. 1. Sudostroyeniye, Leningrad, 1985.
4. HONG YOUNG SUCK, Numerical calculation of second-order wave resistance. *J. Ship Res.* **21,** 2, 94–106, 1977.
5. BONMARIN P., Geometric properties of deep-water breaking waves. *J. Fluid Mech.* **209,** 405–433, 1989.
6. AMROMIN E. L. and TIMOSHIN Yu. S., An algorithm and program for calculating the wave resistance of a catamaran. In *Problems of Ship Construction*, Ser. Mat. Metody, No. 5, 57–60, 1974.
7. AMROMIN E. L., VAL'DMAN N. A. and IVANOV A. N., The non-linear interference of waves from sources and sinks around which there is a flow under the surface of a heavy fluid. In *Asymptotic Methods in Systems Theory*, Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR, 157–164, 1989.
8. GOGISH L. V. and STEPANOV G. Yu., *Turbulent Separated Flows.* Nauka, Moscow, 1979.
9. SHAHSHAHAN A. and LANDWEBER L., Boundary-layer effects on wave resistance of a ship. *J. Ship Res.* **34,** 1, 29–37, 1990.
10. GRIGORYAN S. S., Mechanics of certain large-scale natural processes. In *Proceedings of the Fifth All-Union Congress on Theoretical and Applied Mechanics*, Nauka, Alma-Ata, 1982.

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THE PERTURBATION METHOD IN A SPATIAL PROBLEM OF THE LINEAR VISCOELASTICITY OF ANISOTROPIC BODIES†

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A generalization of an asymptotic method, which has been developed earlier [1, 2] as applied to elastic materials, to the case of viscoelastic anisotropic media is proposed. The problem of the transmission of a load to viscoelastic orthotropic bodies by elastic elements, which is associated with the adhesive strength of composite fibre materials, is investigated.

1. CONSIDER a viscoelastic body consisting of a material which is orthotropic both with respect to its elastic and its viscoelastic properties. The principal directions of anisotropy coincide with the Cartesian axes of the x, y, and z coordinates. In this case, the relationships between the strains and the stresses can be written in the following manner:

$$e_{11} = S_1 - \nu_{12}S_2 - \nu_{13}S_3$$
$$S_i = \frac{1}{E_i}\left(\sigma_{ii} + \int_0^t K_{ii}(t-\tau)\,\sigma_{ii}\,d\tau\right), \quad i = 1, 2, 3 \tag{1.1}$$
$$e_{ij} = \frac{1}{G_{ij}}\left(\sigma_{ij} + \int_0^t K_n(t-\tau)\,\sigma_{ij}\,d\tau\right)$$
$$(i = 2,\ j = 3,\ n = 1;\ i = 1,\ j = 3,\ n = 2;\ i = 1,\ j = 2,\ n = 3)$$

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In order to obtain e_{22} and e_{33}, it is necessary to carry out a cyclic permutation of the indices in e_{11}. Here

$$\nu_{12}E_1 = \nu_{21}E_2,\quad \nu_{23}E_2 = \nu_{32}E_3,\quad \nu_{31}E_3 = \nu_{13}E_1$$
$$K_{12} = K_{21},\quad K_{23} = K_{32},\quad K_{31} = K_{13}$$

Here E are the instantaneous elastic moduli, ν_{ij} are Poisson's ratios, G_{ij} are the shear moduli, σ_{ij} are the normal stresses, $\sigma_{12}=\sigma_{21}$, $\sigma_{13}=\sigma_{31}$, $\sigma_{23}=\sigma_{32}$ are the shear stresses, and $K_{ij}(t-\tau)$ is the creep kernel, for which we use the following analytical expressions [3]:

$$K_{ij}(t-\tau) = k_{ij}(t-\tau)^{\alpha_{ij}-1}\exp[-\beta_{ij}(t-\tau)] \tag{1.2}$$
$$K_i(t-\tau) = k_i(t-\tau)^{\alpha_i-1}\exp[-\beta_i(t-\tau)]\quad (0<\alpha_{ij},\ \alpha_i\leqslant 1)$$

The components of the strain tensor are expressed in terms of projections of the displacements according to the formulas

$$e_{11} = u_x,\ e_{22} = v_y,\ e_{33} = w_z;\ e_{23} = v_z + w_y \tag{1.3}$$
$$e_{13} = u_z + w_x,\ e_{12} = u_y + v_x$$

The indices x, y and z denote differentiation with respect to the corresponding coordinates.

By applying a Laplace transformation with respect to time and with a parameter to relationships (1.1) and taking account of (1.3), we arrive at the integration of a system of equations in the transform of the components of the displacement vector:

$$\begin{gathered}U_{xx} + \varepsilon U_{yy} + \varepsilon l_1 U_{zz} + \varepsilon m V_{xy} + \varepsilon m_1 l_1 W_{xz} = 0\\ \varepsilon V_{xx} + qV_{yy} + \varepsilon l_2 V_{zz} + \varepsilon m_2 U_{xy} + \varepsilon m_3 l_2 W_{yz} = 0\\ \varepsilon l_1 W_{xx} + \varepsilon l_2 W_{yy} + q_1 W_{zz} + \varepsilon m_4 l_1 U_{xz} + \varepsilon m_5 l_2 V_{yz} = 0\end{gathered} \tag{1.4}$$

$$q = \frac{E_2F_{22}(p)}{E_1F_{11}(p)},\quad q_1 = \frac{E_3F_{33}(p)}{E_1F_{11}(p)},\quad \varepsilon = \varepsilon_1\frac{F_3(p)}{F_{11}(p)},\quad \varepsilon_1 = \frac{G_{12}}{E_1},\quad l_1 = \frac{G_{13}F_2(p)}{G_{12}F_3(p)}$$
$$l_2 = G_{23}F_1(p)/G_{12}F_3(p),\ m = 1+\mu,\ m_i = 1+\mu_i,\ \mu = \nu_{12}\varepsilon^{-1}F_{12}(p)$$
$$\mu_1 = \nu_{13}F_{13}(p)(\varepsilon l_1)^{-1},\ \mu_2 = \nu_{21}F_{21}(p)q\varepsilon^{-1},\ \mu_3 = \nu_{23}F_{23}(p)q(\varepsilon l_2)^{-1}$$
$$\mu_4 = \nu_{31}F_{31}(p)q_1(\varepsilon l_1)^{-1},\ \mu_5 = \nu_{32}F_{32}(p)q$$

with the corresponding boundary conditions.

Here,

$$F_{ii}(p) = [1+f_{ii}(p)]^{-1},\ F_{ij}(p) = [1+f_{ij}(p)]F_{jj}(p)\ (i\neq j)$$
$$F_i(p) = [1+f_i(p)]^{-1},\ f_{ij}(p) = k_{ij}\Gamma(\alpha_{ij})(p+\beta_{ij})^{-\alpha_{ij}}$$
$$f_i(p) = k_i\Gamma(\alpha_i)(p+\beta_i)^{-\alpha_i}\ (i, j = 1, 2, 3)$$

Equations (1.4) are analogous to the equilibrium equations in the displacements of an elastic orthotropic body. The asymptotic method in [2] can therefore also be used here if the quantity ε is used as the small parameter. In fact, it turns out to be small since the parameter ε_1 is small in the case of real orthotropic materials and the functions $F_3(p)/F_{11}(p)$ for the kernels (1.2) do not exceed unity for arbitrary values of the parameter p. Here, as in [2], the stress–strain state of a visco-elastic orthotropic body can be successfully split into three components, the determination of each of which reduces to the successive solution of problems in potential theory. This enables us to investigate many problems of the mechanics of a deformed solid which cannot be successfully solved using other methods.

2. We will now consider some problems associated with the determination of the adhesive strength of a fibre composite.

Let an elastic semi-infinite rod with a rectangular cross-section of area F_C (the thickness of the rod is h, the half-width is b and it is assumed that $h/b\ll 1$) be placed in a viscoelastic orthotropic semi-infinite body and be continuously bound to it. The centre line of the insert is perpendicular to the bounding half-space of the plane and coincides with the x axis. We need to determine the law governing the distribution of the contact stresses between the rod and the half-space when a concentrated force P_0 acts at the end point of the rod. This force is directed along the axis of the rod, it is applied at the initial instant of time and subsequently remains constant.

In spatial problems for bodies with elastic inserts of small transverse cross-section, the model of a one-dimensional elastic continuum for the insert in combination with a model of contact along a line is not directly applicable [4]. As in [2, 4], we shall assume that a model of a one-dimensional elastic insert in conjunction with a model of contact over an area for the half-space holds, when the distribution of the contact stresses is given by the formula

$$\sigma_{12}(x, z) = \tau(x)/(\pi\sqrt{b^2 - z^2})$$

where $\tau(x)$ is the stress per unit length of the insert and is to be determined.

In this formulation after the application of a Laplace transformation (the transforms are denoted by asterisks), we arrive at the integration of an equation in the transform of the displacements U_1 of the points of the centre line of the rod

$$U_{1xx} = [P_0\delta(x)/p - \tau^*(x)]/(E_cF_c) \tag{2.1}$$

and the equilibrium equations for the half-space ($q = q_1$, $l_1 = l_2 = 1$)

$$\omega^2U_{xx} + U_{yy} + U_{zz} = 0 \quad (\omega^2 = s^{-1}) \tag{2.2}$$

with the following boundary conditions:

$$\begin{gathered} U_x = 0 \text{ when } x = 0 \\ U_y = \tau^*(x)/(2\pi GF_1(p)\sqrt{b^2 - z^2}) \text{ when } y = 0,\ |z| < b \\ U = U_1 \text{ при } y = 0,\ z = 0 \end{gathered} \tag{2.3}$$

All the functions vanish at infinity.

Here E_c is the modulus of elasticity of the insert material and $\delta(x)$ is the Dirac δ-function. The shear stress $\sigma_{12}^*(x, z)$ is solely determined by the function U_y since $V = 0$ ($V_x = 0$) when $y = 0$ and $\sigma_{12}^* = 2GF_1(p)U_y$, where $G = G_{12}$.

The solution of problem (2.1)–(2.3) can be obtained using Fourier transformations. On carrying out these transformations and finding their inverses, we get

$$U_1(x) = -\frac{2}{\pi}\frac{P_0}{pE_cF_c}\int_0^\infty \frac{M(\theta)\cos xs}{\Delta}ds$$

$$N^*(x) = \frac{2}{\pi}\frac{P_0}{p}\int_0^\infty \frac{sM(\theta)\sin xs}{\Delta}ds, \quad \tau^*(x) = \frac{2P_0\varphi(p)}{\pi p}\int_0^\infty \frac{\cos xs}{\Delta}ds \tag{2.4}$$

$$\begin{gathered} \Delta = s^2M(\theta) + \varphi(p),\ \varphi(p) = \varphi_0F_1(p) \\ \varphi_0 = 2\pi G/(E_cF_c),\ M(\theta) = I_0(\theta)K_0(\theta),\ \theta = b\omega s/2 \end{gathered}$$

where $I_0(\theta)$ and $K_0(\theta)$ are modified Bessel functions and $N^*(x)$ is the Laplace transformation of the stress in the rod.

The inverse Laplace transformation determines the stress N and τ as a function of the coordinates and time. In order to change to the inverse transforms, we represent the stresses (2.4) in the form of series in a small parameter ε_* which depends on p:

$$T^*(x, p) = [T_0(x) + T_1(x)\varepsilon_* + T_2(x)\varepsilon_*^2 + \ldots]/p \tag{2.5}$$

where we mean by T^* either the stress N^* or τ^*.

If the material of the half-space possesses largely shear creep ($k_{ij} = 0$, $k_i = k$) and $\alpha = 1$, then

$$\begin{gathered} \omega = \omega_0[(p + \beta + k)/(p + \beta)]^{1/2},\ \varphi = \varphi_0(p + \beta)/(p + \beta + k) \\ \omega_0 = (E_1/G)^{1/2},\ \theta_0 = b\omega_0s/2 \end{gathered}$$

In this case, $\varepsilon_* = k/(p+\beta)$ in series (2.5) for large values of the parameter p while $\varepsilon_* = \psi p(p+\beta)$, $\psi = -k/(\beta+k)$. On passing to the inverse transform in (2.5), for small values of the time we get

$$T(x, t) = T_0(x) + T_{10}(x)(k/\beta)(1 - e^{-\beta t}) + \ldots \tag{2.6}$$

where, for the stress in the rod $N(x, t)$:

$$T_0(x) = \frac{2P_0}{\pi}\int_0^\infty sM(\theta_0)\frac{\sin xs}{\Delta_0}ds$$

$$T_{10}(x) = \frac{2P_0\varphi_0}{\pi}\int_0^\infty s[M_1(s) - M(\theta_0)]\frac{\sin xs}{\Delta_0^2}ds \tag{2.7}$$

$$\Delta_0 = s^2M(\theta_0) + \varphi_0$$

and for the contact stress $\tau(x, t)$:

$$T_0(x) = \frac{2P_0\varphi_0}{\pi}\int_0^\infty \frac{\cos xs}{\Delta_0}ds$$

$$T_{10}(x) = \frac{2P_0\varphi_0}{\pi}\int_0^\infty s^2[M_1(s) - M(\theta_0)]\frac{\cos xs}{\Delta_0^2}ds \tag{2.8}$$

$$M_1(s) = (\theta_0/4)[I_1(\theta_0)K_0(\theta_0) - I_0(\theta_0)K_1(\theta_0)]$$

For large values of the time, we obtain from (2.5)

$$T(x, t) = T_\infty(x) + T_{1\infty}(x)\psi e^{-\beta t} + \ldots \tag{2.9}$$

The coefficients $T_\infty(x)$ and $T_{1\infty}(x)$ are found using the same formulas (2.7) and (2.8) after replacing ω_0 by ω_∞ and φ_0 by φ_∞, where

$$\omega_\infty = \omega_0(1 + k/\beta)^{1/2},\quad \varphi_\infty = \varphi_0/(1 + k/\beta)$$

The stresses (2.6) can be represented by asymptotic expressions at small and large values of the coordinate x in the same way as in the elastic problem in [2]. For small values of x (which corresponds to a large value of the parameter s), we conclude by using the formulas

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}},\quad K_0(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}$$

that passing to the inverse transforms leads, for small time values, to expansion (2.6), the coefficients of which have the form:

for the stresses in the rod:

$$T_0(x) = 2P_0(\mathrm{ci}\,x_1 \sin x_1 - \cos x_1\,\mathrm{si}\,x_1)/\pi$$

$$T_{10}(x) = -2P_0x_1(\cos x_1\,\mathrm{ci}\,x_1 + \sin x_1\,\mathrm{si}\,x_1)/\pi \tag{2.10}$$

$$x_1 = g_0x,\quad g_0 = b\omega_0\varphi_0$$

and for the contact interaction stress:

$$T_0(x) = -2P_0g_0(\cos x_1\,\mathrm{ci}\,x_1 + \sin x_1\,\mathrm{si}\,x_1)/\pi$$

$$T_{10}(x) = P_0g_0[(\cos x_1\,\mathrm{ci}\,x_1 + \sin x_1\,\mathrm{si}\,x_1) - x_1(\sin x_1\,\mathrm{ci}\,x_1 - \cos x_1\,\mathrm{si}\,x_1 - 1/x_1)]/\pi \tag{2.11}$$

For large values of the time we have the expansions (2.9), the coefficients of which are found using formulas (2.10) and (2.11) after replacing g_0 by $g_\infty = b\omega_\infty\varphi_\infty$. The nature of the decrease in the stresses at large values of the coordinate x is analogous to the elastic problem in [2].

A two-point Padé approximation [5] can be used in order to obtain the inverse transforms of the required functions at arbitrary time values. The process of constructing such an approximation has been described in [6]. In fact, this function enables one to find some characteristics for arbitrary time values, if their behaviour is known for small and large values. The latter, as is shown above, can be determined quite simply.

3. Let thin elastic inserts occupy each of the bands $0 \leqslant x \leqslant \infty$, $|z| \leqslant b$, $y = 2ak$ ($k = 0, \pm1, \ldots$) in the half-space (there is periodicity with respect to the y coordinate). Under the same assumptions as in Sec. 2, the transform of the contact stress is given by the formula

$$\tau^*(x) = \frac{2P_0\varphi}{\pi p}\int_0^\infty \frac{\cos xs}{\varphi + s^2L(s)}ds \tag{3.1}$$

$$L(s) = \int_0^\infty J_0(bv)\frac{\operatorname{cth}\Omega}{\Omega}dv,\quad \Omega = \sqrt{\omega^2s^2 - v^2}$$

When $a \to \infty$ ($2a$ is the distance between the inserts), Eq. (3.1) reduces to the solution for a single insert [the third formula of (2.4)]. The passage to the inverse transforms is accomplished using the method indicated above.

If, in a periodic problem, the rods are loaded through one, the inverse transforms of the contact stress τ_0^* (τ_1^*) in the band connecting the loaded (unloaded) insert with the half-space are written thus:

$$\tau^*_0(x) = \frac{4P_0\varphi}{\pi p}\int_0^\infty \frac{\Delta_2 \cos xs}{\Delta_2^2 - s^4N_1^2(s)}\,ds$$

$$\tau_1^*(x) = \frac{4P_0\varphi}{\pi p}\int_0^\infty \frac{s^2N_1(s)\cos xs}{\Delta_2^2 - s^4N_1^2(s)}\,ds \tag{3.2}$$

$$M_2(s) = \int_0^\infty \frac{J_0(bv)}{\Omega}[\operatorname{th}(a\Omega) + \operatorname{cth}(a\Omega)]\,dv, \quad N_1(s) = 2\int_0^\infty \frac{J_0(bv)\,dv}{\Omega\,\operatorname{sh}(2a\Omega)}$$

$$\Delta_2 = s^2M_2(s) + 2\varphi$$

In this case, the passage to the limit as $a \to \infty$ also yields the solution for a single insert. The contact stresses when $t = 0$ and $t = \infty$ using the same formulas (3.2) after replacing φ by φ_0, ω by ω_0, φ by φ_∞ and ω by ω_∞, respectively, while $p = 1$ also.

It should be noted that the solutions which have been obtained for the contact interaction stresses have a logarithmic singularity at $x = 0$. Actually, the exact nature of the singularity in the neighbourhood of this point has the form

$$\tau(x) = Ax^{-\lambda} \tag{3.3}$$

where λ is known ($0<\lambda<1$) but the coefficient A is unknown. It can be found from the matching conditions: at a certain point of the contact region, both the approximate solutions obtained above and the particular solution (3.3) as well as their derivatives must be identical. These conditions enable one to find the zones in which the particular solutions (3.3) and the solution found by the method proposed above hold.

REFERENCES

1. MANEVICH L. I., PAVLENKO A. V. and KOBLIK S. G., *Asymptotic Methods in the Theory of the Elasticity of an Orthotropic Body*. Vishcha Shkola, Kiev, Donetsk, 1982.
2. PAVLENKO A. V., Application of the asymptotic method to a spatial problem in the theory of elasticity for composite materials. *Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela* 3, 50–61, 1980.
3. GALIN L. A., *Contact Problems in the Theory of Elasticity and Visco-elasticity*. Nauka, Moscow, 1980.
4. ARUTYUNYAN N. Kh. and MKHITARYAN S. M., Some contact problems for a half-space reinforced with elastic coverings. *Prikl. Mat. Mekh.* **36**, 5, 770–787, 1972.
5. BAKER G. and GRAVES-MORRIS P., *Padé Approximations*. Mir, Moscow, 1986.
6. KAGADII T. S. and PAVLENKO A. V., The axially-symmetric problem of the transmission of a load by an elastic rod to a viscoelastic orthotropic body. *Prikl. Mat. Mekh.* **53**, 5, 787–790, 1989.

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